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MESOPOTAMIAN MATHEMATICS

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The term "Mesopotamian mathematics" refers to the mathematical knowledge and the mathematically based practices of the cuneiform tradition from the mid-fourth millennium BCE until its disappearance around the beginning of the Common Era.[1] All dates in the following should thus be understood to be BCE (and according to the "middle chronology") when CE is not indicated explicitly.

The reference to the writing system is not peripheral. Throughout its history, the development and orientation of Mesopotamian mathematics was intimately bound up with written administration and the scribal craft, and all documentation we possess derives from documents written on clay tablets (here, as mostly, the mathematical regularities of buildings and other artifacts give little conclusive information about the kind of mathematical *knowledge* which was involved in their production).

Attentive reading of the written sources reveals, however, that the written tradition must have received important inspiration from traditions carried by non-scribal (and, at least until the advent of Aramaic alphabetic literacy in the first millennium, scarcely literate) specialists: surveyors, master-builders, traders, and/or similar groups. In all likelihood, these "lay" practitioners also borrowed from the literate tradition, but this is more difficult to document.

[1] The changing approach to the field and the increasing awareness that Mesopotamian mathematics has *a history* is described in Jens Høyrup, "Changing Trends in the Historiography of Mesopotamian Mathematics: An Insider's View," *History of Science* 34 (1996), 1–32. An exhaustive annotated bibliography until 1982 is Jöran Friberg, "A Survey of Publications on Sumero-Akkadian Mathematics, Metrology and Related Matters (1854–1982)," *Department of Mathematics, Chalmers University of Technology and the University of Göteborg* 17 (1982). A very detailed account of mathematical knowledge and techniques is Jöran Friberg, "Mathematik," in *Reallexikon der Assyriologie und Vorderasiatischen Archäologie*, vol. 7 (Berlin and New York: de Gruyter, 1990), pp. 531–85. Eleanor Robson has published *Mathematics in Ancient Iraq: A Social History* (Princeton, NJ and Oxford: Princeton University Press, 2008).

LONG-TERM DEVELOPMENTS

So-called protoliterate writing was created in Southern Mesopotamia (the later "Sumerian" area) after the mid-fourth millennium, in connection with the earliest formation of a bureaucratic state (understood as a social system characterized by an at least three-tiered system of control and by extensive specialization of social roles) headed by a temple institution. The root of the invention was an accounting system based on clay tokens (probably standing for various measures of grain, heads of livestock, etc.) that had been used in the Near East since the eighth millennium, and various transformations and extensions of this system introduced in response to the needs created by increasing social complexity.[2]

In the protoliterate period, metrological notations were created that depicted the traditional tokens. A notation for an "almost abstract" number may have been created by adaptation of the system of grain or hollow measures to existing spoken numbers, with basic signs for 1, 10, 60, and 3,600, and composite signs for 60·10 and 3,600·10. Sub-unit extensions of all metrologies, an administrative calendar, and a combined metrology for length and area measurement replacing older "natural" (ploughing or irrigation) measures may also be new creations.[3]

Mathematics was fully integrated with its bureaucratic applications – school texts are "model documents," distinguishable from real administrative documents only by lacking the name of a responsible official and by the prominence of nice numbers. But the integration was mutual: bureaucratic procedures, centered on accounting, were mathematically planned, for instance around the new area metrology and the calendar. The cognitive integration corresponds to social integration – the literate and numerate class seems to coincide with the stratum of temple managers.

The third millennium continued the mutual fecundation of administrative procedures and the development of mathematics (in a process whose details we are unable to follow). The reach of accounting systems increased gradually, and metrologies were modified intentionally so as to facilitate managerial planning and accounting. At the same time, there is a trend toward "sexagesimalization," expanding use of the metrological step factor 60 – see below, "Numbers, Number Systems, Tables, and their Computational Use."

[2] On the token system and its development, see for instance Denise Schmandt-Besserat, *Before Writing.* I. *From Counting to Cuneiform* (Austin, TX: University of Texas Press, 1992).

[3] A broad summary of fourth- and third-millennium mathematical techniques (including the details of metrologies) is Hans J. Nissen, Peter Damerow, and Robert Englund, *Archaic Bookkeeping: Writing and Techniques of Economic Administration in the Ancient Near East* (Chicago, IL: Chicago University Press, 1993). The interplay between state formation and the shaping of mathematical techniques and thought is analyzed in Jens Høyrup, *In Measure, Number, and Weight. Studies in Mathematics and Culture* (New York: State University of New York Press, 1994), pp. 52–7, 68–74. Pages 45–87 of the same volume may serve as a general reference (with extensive bibliography) for the links between statal bureaucracy, scribal craft and culture, and the transformations of mathematics until the mid-second millennium.

Around 2600, however, when a distinct scribal profession emerged, numeracy and literacy outgrew the full cognitive subservience to accounting and management. For the first time, writing served to record literary texts (proverbs, hymns, and epics); and we find the first instances of "pure" or (better) "supra-utilitarian" mathematics – mathematics starting from applicable mathematics but going beyond its usual limits. It seems as if the new class of professional intellectuals set out to test the potentialities of the professional tools – the absolute favorite problem was the division of very large round numbers by divisors that were more difficult than those handled in normal practice.[4]

The language of the protoliterate texts is unidentified, whereas the language of the third-millennium southern city states was certainly Sumerian. Toward 2300, however, an Akkadian-speaking dynasty conquered the whole Sumerian region, and soon, for a while, the entire Syro-Iraqian area (Akkadian is a Semitic language, later split into the Babylonian and Assyrian dialects; names show it to have been present in the area at least since 2600). Sumerian remained the administrative language (and hence the language of scribal education), but new problem types suggest inspiration from a lay, possibly non-Sumerian surveyors' tradition – area computations that are very tedious unless one knows that $\square(R-r) = \square(R) + \square(r) - 2\sqsubset\!\sqsupset(R, r)$ ($\sqsubset\!\sqsupset$ and $\square$ stand for rectangle and square, respectively), and the bisection of a trapezium by means of a parallel transversal.

The twenty-first century is of particular importance. After a breakdown of the "Old Akkadian" empire, a new territorial state ("neo-Sumerian" or "Third Dynasty of Ur") established itself in 2112. A military reform under king Šulgi in 2074 was followed immediately by an administrative reform, in which scribal overseers were made accountable for the outcome of every $^1/_{60}$ of a working day of the labor force allotted to them according to fixed norms; at least in the South, the majority of the working population was apparently subjected to this regime, probably the most meticulous large-scale bureaucracy that ever existed.

Several mathematical tools were apparently developed in connection with the implementation of the reform (all evidence is indirect): a new book-keeping system – not double-entry book-keeping, but provided with similar built-in controls; a place-value system with base 60 used in intermediate calculations; and the various mathematical and technical tables needed in order to make the place-value system useful (described below, p. 64).

4 Two specimens with divisor 7 are analyzed in Jens Høyrup, "Investigations of an Early Sumerian Division Problem, c. 2500 BC," *Historia Mathematica* 9 (1982), 19–36. A similar problem with divisor 33 from Ebla in Syria (whose mathematics was borrowed from Sumer) is analyzed in Jöran Friberg, "The Early Roots of Babylonian Mathematics. III: Three Remarkable Texts from Ancient Ebla," *Vicino Oriente* 6 (1986), 3–25.

No space seems to have been left to autonomous interest in mathematics; once again, the only mathematical school texts we know are "model documents."

For several reasons (among which were probably the exorbitant costs of the administration) even the Ur-III state collapsed around 2000. A number of smaller states arose in the beginning of the succeeding "Old Babylonian" period (2000 to 1600), all to be conquered by Hammurapi around 1760. Without being a genuine market economy, the new social system left much space to individualism, on the socio-economic as well as the ideological level. In the domain of scribal culture, this individualism expressed itself in the ideal of "humanism" (*sic* – n a m-l ú-u l ù, Sumerian for "being human"): scribal *virtuosity* beyond what was needed in practice. This involved the ability to read and speak Sumerian, now a dead language known only by scribes, as well as supra-utilitarian mathematical competence.

The vast majority of Mesopotamian mathematical texts come from the Old Babylonian school (teachers' texts or copies from these, except for the training of simple calculation and copies of tables not student production as all third-millennium specimens). They are in Akkadian (notwithstanding sometimes heavy use of Sumerian word signs), another indication that the whole genre of "humanist" mathematics had no Ur-III antecedents. Its central discipline was a geometrically based second-degree "algebra," probably inspired from a collection of geometrical riddles circulating among lay, Akkadian-speaking surveyors (to find the side of a square from [the sum of] "the side and the area" or from "all four sides and the area," etc.), but transformed into a genuine mathematical discipline and a general analytical technique.

A first classification divides the text corpus into table texts and problem texts. The second category can be subdivided in different ways: into (1) theme texts, whose problems have a common theme; (2) anthology texts which have no common theme; and (3) single-problem texts. Alternatively it can be subdivided into (I) procedure texts that teach how to obtain a solution; and (II) catalog texts listing mere problem statements (most catalogs are theme texts). It is noteworthy that anthology texts, even if mixing different kinds of mathematics, do not mix mathematics with other topics (not even sacred numerology); Old Babylonian mathematics was clearly a cognitively autonomous field.

Some of the texts come from excavations, but most from illegal diggings.[5] For these, provenience and dating must be derived from paleography, orthography, and characteristic differences in terminology. In a region

[5] The basic text editions are O. Neugebauer, *Mathematische Keilschrift-Texte*, 3 vols. (Berlin: Julius Springer, 1935–7) = MKT; O. Neugebauer and A. Sachs, *Mathematical Cuneiform Texts* (New Haven, CT: American Oriental Society, 1945) = MCT; and E. M. Bruins and M. Rutten, *Textes mathématiques de Suse* (Paris: Paul Geuthner, 1961) = TMS. MKT and MCT are very careful editions, TMS alas not. Only TMS contains archaeologically excavated texts. Single texts with

encompassing the former Sumerian South, the Center (Babylon and surroundings) the Center-to-North-East (Ešnunna), and even the eastern periphery (Iranian Susa), the global character of Old Babylonian mathematics is largely the same (from the Assyrian North, never dominated by Ur III or Babylonia, no mathematical texts but only accounts are known). Close attention to language and procedures reveals, however: that the adoption of lay material has taken place simultaneously in Ešnunna and in the South; that pre- and post-Šulgi-reform Sumerian mathematics coexisted in eighteenth-century Ešnunna without being fully merged; that a number of schools tried to develop a strict terminological canon but did not agree in their choices; and that all texts that try to explain procedures abstractly and not only through paradigmatic numerical examples are close to the lay oral tradition – the school seems to have given up abstract formulation as pedagogically inefficient.[6]

Inner weakening followed by a Hittite raid put an end to the Old Babylonian state in 1600. A warrior tribe (the Kassites) subdued the Babylonian area, for the first time rejecting that managerial-functional legitimization of the state which, irrespective of suppressive realities, had survived since the protoliterate phase and made mathematical-administrative activity an important ingredient of scribal professional pride. The school institution disappeared, and scribes were trained henceforth as apprectices. Together, these events had the effect that mathematics disappears almost completely from the archaeological horizon for a millennium or more (one Kasssite problem text and one table text have been found; the problem text offering a sham solution to a very difficult problem seems to derive from the style of the Old Babylonian northern periphery); metrologies were modified in a way that would fit practical computation in a mathematically less sophisticated environment (e.g. making use of normalized seed measures in area mensuration; though no longer an object of pride, mathematical administration did not disappear).

Around the "Neo-Babylonian" mid-first millennium, mathematical texts turn up again, for instance concerned with area mensuration, the conversion between various seed measures, and some supra-utilitarian problems of the kind that had once inspired Old Babylonian "algebra." This and other features may reflect renewed interaction between the scribal and the lay traditions, which so far cannot be traced more precisely.

known provenience have been published by Taha Baqir, Jöran Friberg, and others, many in the journal *Sumer*.

6 The analysis is presented in Jens Høyrup, "The Finer Structure of the Old Babylonian Mathematical Corpus. Elements of Classification, with some Results," in Joachim Marzahn and Hans Neumann (eds.), *Assyriologica et Semitica. Festschrift für Joachim Oelsner anläßlich seines 65. Geburtstages am 18. Februar 1997* (Münster: Ugarit Verlag, 2000), pp. 117–77. The idea that traces of pre-Šulgi mathematics might be present in texts from the northern periphery was first proposed by Eleanor Robson in her dissertation from 1995, now published as *Mesopotamian Mathematics 2100–1600* BC. *Technical Constants in Bureaucracy and Education* (Oxford: Clarendon Press, 1999). The precise chronology for the adoption of various kinds of lay material, as far as it can be known, is investigated in Jens Høyrup, "A Hypothetical History of Old Babylonian Mathematics: Places, Passages, Stages, Development," *Gaṇita Bhārati* 34 (2012), 1–23.

One Neo-Babylonian text combines the sacred numbers of the gods with a metrological table. This breakdown of cognitive autonomy corresponds to what the texts tell us about their owners and producers (such information is absent from the Old Babylonian tablets); they identify themselves as "exorcists" or "omen priests" (another reason to believe that their practical geometry was borrowed from lay surveyors).

A final development took place in the Seleucid era (311 onwards). Even this phase is only documented by a few texts: some multi-place tables of reciprocals possibly connected to astronomical computation, though of no direct technical relevance; one theme text; an anthology text focusing on practical geometry; and an unfocused anthology text. The unfocused anthology text shows some continuity with the Old Babylonian tradition (including its second-degree "algebra") but also fresh developments (e.g., formulas for $\sum 2^n$ and $\sum n^2$). The theme text contains "algebraic" problems about rectangles and their diagonals, of which only one type is known from the earlier record, but where even this is solved in a different way. It seems to be a list of *new* problem types or procedures, either borrowed from elsewhere or freshly invented. The Seleucid texts make heavy use of Sumerian word signs, but in a way that sometimes directly contradicts earlier uses. To some extent at least they represent a new translation into Sumerian of a tradition that must have been transmitted outside an erudite scribal environment.

In connection with the creation of a planetary astronomy based on arithmetical schemes, the Neo-Babylonian period (in particular the Seleucid phase) developed a set of highly sophisticated numerical techniques; these are dealt with in chapter 4 by John Steele, in this volume.

NUMBERS, NUMBER SYSTEMS, TABLES, AND THEIR COMPUTATIONAL USE

The original number system was based on specific signs for 1, 10, 60, 600, 3,600, and 36,000, multiples of which were produced by repetition in fixed patterns (· , ·· , ··· , :: , ·.·.· , etc.). In the third millennium, adjunction of the sign g a l, "great," allowed upwards extension of the system by a factor 60, whereas the sign gín, borrowed from weight metrology, was used in the sense of $^1/_{60}$ (the same tricks were used to expand the reach of metrological sequences). From the Old Akkadian epoch onward, calculators can be seen to have experimented with the system, thus approaching the place-value principle – but all the relevant texts commit errors, thus showing that no place-value *system* was yet available.[7]

[7] See Marvin A. Powell, "The Antecedents of Old Babylonian Place Notation and the Early History of Babylonian Mathematics," *Historia Mathematica* 3 (1976), 417–39.

The *system* seems to have been created in the wake of the Ur-III administrative reform. It employed the traditional sign for 1 for any integer power 60^n, and the sign for 10 for any $10 \cdot 60^n$ – still with repetitions in fixed patterns to express 2, 3, . . .,9, and 20, 30, . . ., 50. It was a floating-point system, with no indication of absolute order of magnitude, as the slide rule engineers would use until some decades ago. Nor were "intermediate zeroes" indicated. For both reasons, the notation could only be used for rough work – final results had to be inserted in the documents in the traditional, unambiguous notation.

The place-value notation did not facilitate additions and subtractions – these were performed on a calculating board;[8] the reason it was introduced was the importance of multiplications in Ur-III planning and accounting. If, e.g., the labor needed to produce a wall of given dimensions of bricks of a given type was to be found, one ("metrological") table would translate a thickness measured in cubits and fingers into the standard length unit (a "rod" ≈ 6m), after which the volume of the wall could be found in standard units. A "technical" table of "constant factors" would tell the number of bricks of the type in question per unit volume, another the number of bricks produced by a worker per day, a third the number carried a given distance per man-day, etc. The total consumption of labor could then be found by means of multiplications and divisions.[9]

Beyond metrological conversion tables and tables of technical constants, the system depended on the availability of multiplication tables and of tables of reciprocals (to be learned by heart in the scribe school) – the latter because division by n was performed as a multiplication by $^1/_n$. The important step in the invention of the place-value system was thus not the inception of the idea, which had been in the air for centuries; it will have been the government decision to have it spread in teaching and to produce (mass-produce!) the tables needed for its implementation.

Once introduced, the place-value system could survive in less bureaucratic settings. It became the standard system of Old Babylonian mathematical texts (only occasionally will the units of "real" life turn up in statements or final results) and of late Babylonian mathematical astronomy. It is quite uncertain whether the Indian decimal place-value system for integers depends on it, but it is undisputed that it was taken over in the minute-second fractions of Greek and later astronomy, whence it inspired the introduction of decimal fractions.

It may seem a drawback of the Babylonian division method that it only works for "regular" divisors of the form $2^p \cdot 3^q \cdot 5^r$ (p, q, and r positive, negative,

[8] See Jens Høyrup, "A Note on Old Babylonian Computational Techniques," *Historia Mathematica* 29 (2002), 193–98, and Christine Proust, "La multiplication babylonienne: la part non écrite du calcul," *Revue d'Histoire des Mathématiques* 6 (2000), 293–303.

[9] The most thorough treatment to date of the technical factors and their use as reflected in mathematical texts is Robson, *Mesopotamian Mathematics 2100–1600 BC*.

or 0). In practice this was no trouble, firstly because all metrological step factors were regular, and secondly because the margin on technical factors was always large enough to allow representation by a simple regular number. Factors that might turn up as divisors were always chosen thus.

Beyond the tables already mentioned, other arithmetical tables occur: n^2 (with inversions as $\sqrt{N}$, where N itself is square), and the inversions of n^3 and $n^2 \cdot (n+1)$. Tables of squares (*viz* square areas expressed in metrological units) go back to before the mid-third millennium and thus antedate the place-value system by ca. 500 years.

GEOMETRY

Very few third-millennium texts reveal the actual mathematical knowledge and procedures that went into their results. The post-Old-Babylonian texts at our disposal are also too few to suggest any global picture. For these reasons, this and the following two sections deal primarily with the mathematics and the mathematical thought of the Old Babylonian period.

In geometry, no concept of the *quantifiable* angle existed. In order to find the area of a rectangle the Babylonians would multiply the length with the width – as mentioned, the area metrology had been adapted to this already in the fourth millennium. When dealing with near-rectangular quadrangles they would choose as length and width the legs of an approximately right angle (as opposed, we may say, to a "wrong" angle). If opposite sides were slightly different, average length would be multiplied by average width (the "surveyors' formula"; also since the fourth millennium). This would always yield too large results, but with one known exception it was only used in practical mensuration when the error was negligible (when used as a mere pretext for supra-utilitarian problems in the Old Babylonian school, the formula might be employed in cases where it is blatantly absurd).

The area of approximately right triangles was found as the product of the bisected width with "the length" – as opposed to "the long length," i.e., the hypotenuse. More complex shapes would be split up into quasi-rectangles and quasi-right triangles (this is seen in Ur-III field plans). A Seleucid text computes the height of an equilateral trapezium; a text from Old Babylonian Susa suggests that the same *could* be done in earlier times when regular polygons were investigated.

The absence of the notion of the quantifiable angle did not prevent the understanding of similarity relations. It was also routinely used that the areas of similar figures are to each other as the squares on the linear dimensions.

It was known that the square on the diagonal of a rectangle augmented by the doubled area equals the square on the sum of the sides, whereas the squared diagonal minus the doubled area equals the square on their difference – and, probably as a sequel, that the squared diagonal itself equals the sum of the squared sides. The latter, of course, is what we know as the "Pythagorean theorem."

The fundamental circle parameter was the perimeter p – the area was found as $^1/_{12}p^2$, and the diameter as $^1/_3p$. In one text group from the northern region (in general close to the lay tradition, where both the separate treatment of the semicircle and the very same formula turn up in later ages) the area of the semicircle is found as $^1/_4$ of the product of diameter and arc.

In volume metrology, the area units were thought of as provided with a "standard thickness" of 1 cubit. In order to determine a prismatic or cylindrical volume, the calculator would first find the base (with this implicit thickness) and then "raise it to," i.e, multiply it with, the height. This operation was so important that "raising" became the standard term for any multiplication which was based on similar considerations of proportionality (only concrete repetition and the laying-out of rectangular areas employ other terms); all multiplications with factors taken from metrological tables or tables of technical constants were thus "raisings."

The volume of a truncated cone was calculated as the height times the mid-cross-section, that is, as that of a cylinder with the average diameter. In one case, the volume of a truncated pyramid is determined as the average base raised to the height – in another, the correct value is found, whether from a correct formula or not is unclear (a correct formula *can* be derived from relatively simple intuitive arguments).

The simple area and volume formulas are without doubt based on such intuitive insights. The restricted use of the "surveyors' formula" indicates that it was understood to be only an approximation, but nothing suggests any precise idea as to the importance of the error; probably the Babylonians would see no difference between this kind of approximation and the treatment of an inevitably uneven terrain as if it were a perfect plane.

Formal demonstration seems to be absent from Babylonian geometry. There was a certain interest in striking geometrical configurations – e.g., systems of concentric squares. Reflections on a concentric two-square system and on the appurtenant "average square" may have led to the discovery of how to bisect a trapezium by a parallel transversal. Apart from this, the only important kind of supra-utilitarian geometry was the area technique which has become known as "Babylonian algebra."

"ALGEBRA" AND OTHER "PURE" PURSUITS

When it was discovered in the late 1920s that the sequence of numbers in certain texts corresponds to the solution of second-degree equations, much of the technical terminology was still uninterpreted. It was assumed – and generally accepted for 60 years – that the underlying conceptualizations were arithmetical; that the operations involved were therefore numerical

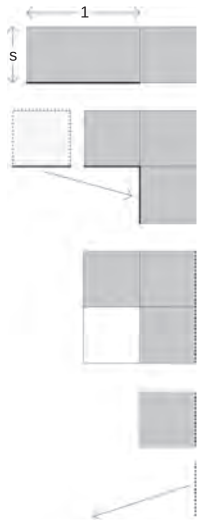

Figure 3.1. The procedure of BM 13901 n° 1.

additions, subtractions, multiplications, and extractions of roots; and that the persistent references to lengths, widths, and areas were nothing but metaphors for numerical unknowns and their products.[10]

Close attention to the vocabulary and the organization of the texts demonstrates, however, that two presumed additions are kept strictly apart; that there are two different subtractive operations; that two different "halves" are distinguished; and that "multiplications" are four in number. All of this concerns *concepts* – several of the concepts are covered by two or more synonymous *terms*. This makes no sense in the arithmetical interpretation, but everything becomes obvious if we take the words of the texts (lengths, widths, squares, areas) seriously. Babylonian "algebra" turns out to be a cut-and-paste technique which manipulates measurable line segments and areas in analytic processes which, in their numerical steps, correspond to the procedures of our equation algebra.[11] As an example we may look at the simplest of all mixed second-degree problems: the sum of a square area and the side is 45′ (i.e., $45 \cdot {}^1/_{60} = {}^3/_4$). The sequence of numerical steps in the solution (with added indications of absolute magnitude, ′ indicating minutes or sixtieths, ′′ seconds) is as follows: 45′ — 1 — 1 — 30′ (= $^1/_2$) — 30′ — 15′ (= $^1/_4$) — 45′ — 1 — 1 — 30′ — 1 — 30′.[12] What goes on can be followed in Figure 3.1. At first the side is represented by

[10] The discovery and the development of interpretations is analyzed in Høyrup, "Changing Trends," pp. 1–10.

[11] See Jens Høyrup, *Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and Its Kin* (Studies and Sources in the History of Mathematics and Physical Sciences; New York: Springer, 2002).

[12] BM 13901 n° 1, in MKT III, 1.

a rectangular area ⊏⊐(1,s), which is glued to the square □(s). Its length 1 is bisected, and the outer $^1/_2$ is moved so as to span with the $^1/_2$ that remains in place a quadratic complement with the area $^1/_4$. This is joined to the gnomonic area $^3/_4$ consisting of the square and the bisected rectangle. The resulting square has the area 1, and thus also the side 1. The $^1/_2$ that was moved is detached from this 1, and $^1/_2$ remains as the side of the original square. The method, as we see, is analytical in the same sense as equation algebra: the unknown side s is treated *as if* it were known, and the complex relation subjected to manipulations until s appears in isolation.

This is one of those original surveyors' riddles that were apparently borrowed by the early Old Babylonian scribe school. In the scribe school it was only one of many problems dealing with areas and segments. In non-normalized cases, a proportional scaling of figures along one dimension was used along with the cut-and-paste procedures.

Line segments and areas constituted the basis of the technique. They could then be used to *represent* entities of other kinds: numbers from the table of reciprocals, prices – or segments might represent areas or volumes. The technique thus served as a general tool for finding unknown entities involved in complex relationships. Even in this sense, it was similar to equation algebra; its "basic representation" was not numerical, it is true, but the segments and areas of this representation were as *functionally abstract* as the numbers of equation algebra. The closest kin of Babylonian "algebra," however, is pre-Viète algebra: it was and remained a *technique*, and was never associated with any algebraic theory about solvability conditions or the classification of problems (classification was based on geometrical object and not on algebraic type, as revealed by the organization of theme texts). Nor was it used to solve "real-life" problems – no single practical problem presenting itself to Babylonian calculators was of the second degree. The only "practical" purpose of treating second-degree problems in school was as a pretext for training calculation with sexagesimal numbers (much as second-degree equation algebra has served in the schools of recent centuries to train the manipulation of algebraic letter symbols).

Practical first-degree problems were solved without recourse to algebraic techniques, often by variants of the "single false position" (also used in homogeneous problems of the second and third degree). However, second-degree "algebraic" systems might include a genuine first-degree equation, of the type "the sum of the length and the width, from which $^1/_4$ of the width is detached, is 45"). A couple of texts discuss such equations and their transformation, identifying most pedagogically the coefficients of length and width and the contribution of each to the sum.

Some higher-degree problems (of biquadratic and similar types) were solved by means of the algebraic technique. Mixed third-degree problems also turn up – e.g., to find the side of a cubic excavation if the sum of the volume and the base is known. Here the algebraic technique would forsake.

Instead the calculator resorted to a combination of false-position considerations and factorization or (in the case just mentioned) the table of $n^2 \cdot (n+1)$. The trick is elegant but *only works because a simple solution is known in advance to exist* (*all* school problems were constructed backwards from known solutions).

The cubic problems are found in theme texts together with other "excavation" problems of the first or the second degree, solved on their part with algebraic methods. As regards their method, however, they are rather linked with another kind of supra-utilitarian mathematics: investigations of the properties of the regular numbers of the sexagesimal place value system. In simple cases, it involved factorizations, continued multiplication products of simple factors, etc. The high point is a tabulation, not directly of Pythagorean triplets *a–b–c* but of ???–z^2–*b*–*c*, where ??? stands for one or more missing columns, and $z=\frac{c}{a}$. All sets $(\overline{b},\overline{c}) = (\frac{t'-t}{2}, \frac{t'+t}{2})$ are listed for which $\sqrt{2}-1<t<{}^5/_9$, t being the quotient between two regular integer numbers no greater that 125, $t'=\frac{1}{t}$.[13]

The headings of the *b*- and *c*-columns speak about width and diagonal, and it is thus certain that a geometric rectangle and its diagonal are involved. Apart from that, the purpose of the text is obscure. As shown by Friberg, it is not the result of a pure number-theoretical investigation. Instead, he proposes, it might serve as a tool for finding an array of data that would permit some mathematical problem (e.g. concerning right triangles) to be solvable. Unfortunately, Old Babylonian problems always have very simple solutions, and consecutive problems often stick to the same solution; available evidence therefore speaks against this proposal, but no more convincing alternative is at hand. The text adds an important shade to our knowledge about *what* the Babylonians could do but so far nothing to our understanding of *why* they would do it.

"MATHEMATICS" OR "COMPUTATION"? A GLOBAL CHARACTERIZATION

In the Old Babylonian period, mathematics was a cognitively autonomous subject, and it may therefore be considered legitimate to speak of it precisely as *mathematics*, as done until now. In contrast, the term "mathematician" appears nowhere. All we know about the authors of the mathematical texts with some certainty (namely from the format of the texts) is that they will have been teachers of future scribes (even though the sophisticated matters were

[13] The tablet has been much discussed in the literature. Analysis and summary of earlier work is found in Jöran Friberg, "Methods and Traditions of Babylonian Mathematics. Plimpton 322, Pythagorean Triples, and the Babylonian Triangle Parameter Equations," *Historia Mathematica* 8 (1981), 277–318. A new profound analysis is Eleanor Robson, "Neither Sherlock Holmes nor Babylon: A Reassessment of Plimpton 322," *Historia Mathematica* 28 (2001), 167–206.

hardly tought to more than a minority of these). Much of what we find in the texts is supra-utilitarian – but its ultimate legitimacy always rests on its link to scribal activity. The scribe, however, when using mathematics, would always be interested in *finding a number*, not, e.g., in geometrical regularities; artists might have this interest, but with the exception of the abovementioned concentric squares nothing permits us to link patterns with mathematically interesting symmetries to the mathematical texts.

Strictly speaking, Old Babylonian (and, in general, Mesopotamian) mathematics might therefore better be characterized as *computation*; instead of "mathematicians" we should speak of "calculators" and "teachers of calculation"; supra-utilitarian activities represent "pure calculation" rather than "pure mathematics." The ultimate interest in finding a number is of course also a characteristic of most present-day applications of mathematics; but it remains a feature which distinguishes both the Mesopotamian and the contemporary calculating orientation from the *investigation of the properties of mathematical objects* which (since the Greeks) constitutes our ideal type of mathematics proper.

The italicized passage contains a veiled reference to another difference between our ideal type and the Mesopotamian type: "investigation." In principle, theoretical mathematics has *the problem* as its core, and then sets out to construct methods and a conceptual apparatus that permit its solution. The same characteristic holds for *applications* of mathematics (Mesopotamian as well as contemporary), with the difference that the defining problem is no mathematical problem. The core of Mesopotamian supra-utilitarian mathematics, on the contrary, was always *the method*. When the mid-third-millennium calculators were testing the potentialities of the professional tools, these tools were the starting point, and the aim was to find out how far they would reach. Similarly, the "scribal humanism" of the Old Babylonian period, aiming at handling with *virtuosity* the tools and techniques of the scribe, would be centered on these.

This does not preclude the practical existence of mathematical research, in the form of a search for problems that could be treated by available techniques and tricks. The difference between the surveyors' riddles and the algebraic discipline created in the school is indeed the outcome of this kind of search. Nor did it preclude the invention of new techniques of scarce practical utility; such inventions might be needed if new problem types were to be transformed so as to be solvable – the "quadratic completion" used to solve mixed quadratic problems is an example, already conceived in the lay surveyors' environment and then adopted into the early Old Babylonian scribe school (which knew it as "the Akkadian [method]"). Once devised, such techniques would themselves become part of the stock of professional tools, and serve in the search for problem types that might now be solved – as the quadratic completion became the basis for the whole fabulous development of second-degree algebra in the school.

REVERBERATIONS

After the discovery of the Babylonian second-degree algebraic in the late 1920s, Neugebauer proposed that the geometry of *Elements* II (characterized by Zeuthen as "geometric algebra" already in the 1880s) should be understood as a geometrical translation of the supposedly numerical algebra of the Babylonians, prompted by the discovery of irrationality and the ensuing "foundation crisis" of Greek mathematics.

The foundation crisis turned out to be a projection of the 1920s on Greek antiquity, and even the translation theory proved problematic as it was formulated. *Elements* II solve no problems, at most they can be said to prove algebraic identities of a kind that Babylonian algebra seemed to be based on. Worse was the disappearance of the main stock of Babylonian algebra more than a millennium before the creation of Greek geometry and the failing evidence that any Greek mathematician knew about Babylonian mathematics.

The geometric reinterpretation of the Babylonian technique transforms the question: Euclid's diagrams coincide with those of which the Babylonians had made use (II.6 thus with the procedure shown above), and his proofs may be said to provide a "critique" of the Babylonian procedures – verification of their legitimacy and investigation of the conditions under which they are valid. But it does not invalidate the second objection to Neugebauer's thesis.

Comparative analysis of the Babylonian material and a number of later sources – mostly treatises on practical geometry containing supra-utilitarian material, many from the Islamic Middle Ages but others belonging to classical antiquity or to the stock of Italian borrowings from lost Arabic sources – now allows us to delineate a new scenario.[14]

The original stock of quasi-algebraic surveyors' riddles can be said with fair certainty to have encompassed at least the following problems on a single square (area A, side s, "all four sides" ${}_4s$; Greek letters indicate given numbers):

$$A \pm s = \alpha \ // \ A + {}_4s = \beta \ // \ A = {}_4s.$$

On rectangles (length l, width w, all sides ${}_4s$, diagonal d) the following can be identified:

[14] The details of the scenario and fairly full arguments from the sources are found in Jens Høyrup, "On a Collection of Geometrical Riddles and Their Role in the Shaping of Four to Six 'Algebras'," *Science in Context* 14 (2001), 85–131. Supplementary material is in "Hero, Ps.-Hero, and Near Eastern Practical Geometry. An Investigation of *Metrica*, *Geometrica*, and other Treatises," in Klaus Döring, Bernhard Herzhoff, and Georg Wöhrle (eds.), *Antike Naturwissenschaft und ihre Rezeption, Band 7* (Trier: Wissenschaftlicher Verlag Trier, 1997), pp. 67–93. The changing and incoherent uses of the notion of "geometric algebra" since Zeuthen by friends and foes are analyzed in Jens Høyrup, "What is 'Geometric Algebra', and What Has It Been in Historiography?" *AIMS Mathematics* 2 (2017), 128–60.

$$A = \alpha, l \pm w = \beta \;//\; A + (l \mp w) = \alpha, l \pm w = \beta \;//\; A = \alpha, d = \beta,$$

and seemingly also

$$A = l + w//A{=}_4s.$$

On two squares, finally,

$$A_1 + A_2 = \alpha, s_1 \pm s_2 = \beta \;//\; A_1\text{–}A_2 = \alpha, s_1 \pm s_2 = \beta.$$

The lay tradition – whose geographical extension may have outranged Mesopotamia – survived the collapse of the Old Babylonian scribe school, and conserved its stock of riddles. It may have borrowed from the scribe school, but only marginally, and never anything "algebraic" of a more advanced character than the original riddles. Some of its characteristic riddles turn up in Diophantus' *Arithmetica* I, some are found in pseudo-Heronian or agrimensor treatises, and some are referred to in the *Theologoumena arithmeticae* – enough, indeed, to demonstrate that Greek theoretical geometry would have had no difficulty in running into the tradition (whether during contacts with Syro-Phoenician practitioners or in Egypt, to where it may have been brought by military surveyors or tax collectors in the wake of the Assyrian or the Persian conquests). It seems that some geometers did so before Theodoros' time (thus probably in the fifth century) and submitted the old procedures to a "critique" whose results turn up in *Elements* II, propositions 1–10; all of these, indeed, are related to the basic riddles or to the formulas $\square(R \pm r) = \square(R) + \square(r) \pm 2\sqsubset\!\sqsupset(R, r)$, apparently known already in the Old Akkadian school. In contrast, nothing in Euclid relates to the particular creations of the Old Babylonian scribe school: ample and intricate use of coefficients; the treatment of biquadratics and other higher-order problems; and the scaling of non-normalized problems (*Elements* VI.28–9 is likely to represent an independent though similar generalization).

In the Islamic world, the tradition and at least some of the riddles are still encountered around 1200 CE. In the ninth century CE, the cut-and-paste procedures were borrowed by al-Khwārizmī for his demonstrations of the algorithms of *al-jabr*, and thus were also adopted as a core constituent of Latin *algebra*.

References to the old tradition are also found in Mahāvīrā's ninth-century *Gaṇita-Sāra-Sangraha*. Since they do not correspond to what occurs in Islamic sources, Mahāvīrā is likely to have drawn on the Jaina tradition. He is thus a witness of a possible link between the Near Eastern tradition and Indian medieval algebra – a link which is invisible in the numerical algebra of Āryabhata and Brahmagupta. If not directly, then at least through this lay tradition, the Babylonian algebra discovered by Neugebauer and his collaborators thus had even wider repercussions than he ever dared imagine in print.